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YOU ARE ALLOWED TO SUBMIT SOLUTIONS TO ONLY FIVE PROBLEMS

EXAM ID NUMBER:________

COURSE NUMBER: AMCS 211

PROBLEM: 1

Principal Component Analysis [20 pts]

In this problem, we will derive a closed form solution to a traditional problem in pattern recognition, called principal component analysis (PCA).

1. To simplify the discussion of PCA, consider the constrained optimization problem in Eq (2), where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric.

$$\mathbf{x}^* = \underset{\mathbf{x}}{\arg\max} \quad f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad \text{subject to: } \|\mathbf{x}\|_2^2 = 1 \tag{2}$$

   (i) [**3 pts**] Write down the KKT conditions for Eq (2).

   (ii) [**3 pts**] Show that $\mathbf{x}^*$ must be an eigenvector of matrix $\mathbf{A}$.

   (iii) [**4 pts**] Show that $f(\mathbf{x}^*) = \sigma_1(\mathbf{A})$, where $\sigma_1(\mathbf{A})$ is the maximum eigenvalue of $\mathbf{A}$ and $\mathbf{x}^*$ is its corresponding eigenvector.

2. Consider a set of m points $\{\mathbf{y}_i\}_{i=1}^m$, where each $\mathbf{y}_i \in \mathbb{R}^n$. We seek the unit norm direction $\mathbf{w} \in \mathbb{R}^n$, along which the empirical variance of the projection of all m points is *maximum*. Here, we define the projection of $\mathbf{y}_i$ along $\mathbf{w}$ as: $p_i = \mathbf{w}^T \mathbf{y}_i$. We can form the empirical variance of this projection as in Eq (3).

$$h(\mathbf{w}) = \frac{1}{m} \sum_{i=1}^{m} (p_i - \bar{p})^2, \quad \text{where } \bar{p} = \frac{1}{m} \sum_{i=1}^{m} p_i \tag{3}$$

   (i) [**6 pts**] Let $\bar{\mathbf{y}} = \frac{1}{m} \sum_{i=1}^m \mathbf{y}_i$ and $\mathbf{z}_i = \mathbf{y}_i - \bar{\mathbf{y}}$. Show that: $h(\mathbf{w}) = \mathbf{w}^T \mathbf{C} \mathbf{w}$, where $\mathbf{C} = \frac{1}{m} \mathbf{Z} \mathbf{Z}^T$ is the covariance matrix of the m points and $\mathbf{z}_i$ is the i^{th} column of matrix $\mathbf{Z} \in \mathbb{R}^{n \times n}$.

   (ii) [**4 pts**] PCA is the process of finding a unit norm vector $\mathbf{w}$ that maximizes the empirical projection variance $h(\mathbf{w})$. Formulate PCA as an optimization problem and use the previous results to find a closed form global solution.

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EXAM ID NUMBER:________

COURSE NUMBER: AMCS 211

PROBLEM: 2

Variable Elimination [20 pts]

Consider the optimization problem in Eq (3).

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} \\ \text{subject to:} \quad & \mathbf{c}^T\mathbf{x} = e \end{aligned} \tag{3}$$

Take $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $e = 2$

1. **[5 pts]** Show that Eq (3) can be formulated as an unconstrained optimization problem using variable elimination. Give a detailed description of how this can be done.
2. **[5 pts]** Show that this unconstrained problem is convex.
3. **[5 pts]** Now, compute the optimal solution $\mathbf{x}^*$ and its corresponding cost function $f(\mathbf{x}^*)$. You can use the following identity for invertible matrices in $\mathbb{R}^{2\times 2}$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4. **[5 pts]** Is your solution $\mathbf{x}^*$ a global minimizer of Eq (3)? Explain why or why not.

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EXAM ID NUMBER:________

COURSE NUMBER: AMCS 241

PROBLEM: 1

AMCS 241

Probability and Random Processes

Problem 1 (20 Points)

Part 1 (10 Points)

Let X be a continuous uniform random variable over $[-1, 1]$. Let $Y = X^n$, where n is a positive integer.

(a) **(5 Points)** Calculate the covariance of X and Y.

(b) **(5 Points)** Calculate the correlation coefficient of X and Y.

Part 2 (10 Points)

Let X be a Gaussian random variable with zero mean and variance σ^2

(a) **(5 Points)** Find the conditional expectation $E[X|X>0]$

(b) **(5 points)** Find the conditional variance $Var[X|X>0]$.

Recall that the probability density function of a Gaussian random variable X is given by

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean and σ is the standard deviation of the random variable X.

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EXAM ID NUMBER:________

COURSE NUMBER: AMCS 241

PROBLEM: 2

AMCS 241

Probability and Random Processes

Problem 2 (20 Points)

Part 1 (10 Points)

Let X and Y be two continuous independent random variables both uniformly distributed over $[0, 1]$. Find the probability density function of $W=X-Y$.

Part 2 (10 Points)

Consider a random process $X(t)$ defined by

$X(t)= U \cos t + (V+1) \sin t$,

where U and V are independent random variables for which $E[U]=E[V]=0$ and $E[U^2]=E[V^2]=1$.

(a) **(5 Points)** Find the auto-covariance function of $X(t)$

(b) **(5 Points)** Is $X(t)$ a wide sense stationary (WSS) random process ?

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EXAM ID NUMBER:________

COURSE NUMBER: EE 242

PROBLEM: 1

PhD Qualifier Exam - Spring 2015
Digital Communications

Question No. 1 (20 marks) A communication system transmits one out of 4 symbols with probabilities according to the table shown below:

Symbol	Prob.	Bits	Energy
s_1	0.4	00	3
s_2	0.1	11	1
s_3	0.2	10	5
s_4	0.3	01	13

a. Determine the average energy of the signal set. (2 marks)

b. Determine the average energy per bit. (2 marks)

Now the symbols are transmitted in a noisy channel and due to noise, the receiver confuses one symbol with another with some probability as shown in table below:

T_x \ R_x	s_1	s_2	s_3	s_4
s_1		0.1	0.01	0.01
s_2	0.1		0.1	0.01
s_3	0.01	0.1		0.2
s_4	0.1	0.1	0.2	

c. Assume that s_1 is transmitted. Determine the probability of error. (4 marks)

d. Determine the probability of symbol error for this system. (5 marks)

e. Determine the bit assignment for the symbols such as to minimize the probability of bit error. Determine the bit error rate for the chosen bit assignment. (7 marks)

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EXAM ID NUMBER: ________

COURSE NUMBER: EE 242

PROBLEM: 2

Question No. 2 (20 marks) Consider the following two signals used in a communications system over an AWGN channel.

$$x_0(t) = \begin{cases} \sqrt{\frac{2\mathcal{E}_x}{T}} \cos(2\pi f_0 t) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t) = \begin{cases} \sqrt{\frac{2\mathcal{E}_x}{T}} \cos(2\pi (f_0 + \Delta) t) & \text{if } t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

$$T = 100\mu s \quad f_0 = 10^5\text{Hz} \quad \sigma^2 = 0.01 \quad \mathcal{E}_x = 0.32$$

a. Find P_e if $\Delta = 10^4$. (5 marks)

b. Find the smallest $|\Delta|$ such that the same P_e found in part (a) is maintained. What type of constellation is this? (5 marks)

c. Find P_e for $\Delta = 10^2$. (5 marks)

d. How can you improve the energy efficiency of the modulation in part (a). (5 marks)

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EXAM ID NUMBER:________

COURSE NUMBER: EE 251

PROBLEM: 1

(1) Assume that we have an *N*-point FFT block, where *N* is even. We know the **outputs** $X(2m), m = 0,1,..-1 + N/2$, and the **inputs** $x(v), v = 0,1,..-1 + N/2$.

(a) Devise a general procedure for computing the remaining outputs? (10 marks)

(b) Numerically solve for the case where $x(0) = 1, x(1) = x(2) = \cdots x\left(-1 + \frac{N}{2}\right) = 0, X(0) = X(2) = \cdots = X(-2 + N) = 2$. (10 marks)

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EXAM ID NUMBER:________

COURSE NUMBER: EE 251

PROBLEM: 2

(2) (a) How can we compute the *N*-point DFT of two **real-valued** sequences using one *N*-point DFT? That is, in your algorithm, you are allowed one and only one *N*-point DFT operation. What is required is the DFT of both real-valued sequences. Note that the DFT accepts complex inputs. (10 marks)

(b) A sleep-deprived PhD student was trying to compute the linear convolution of two discrete-time sequences $x_1(n)$ of length N_1 and $x_2(n)$ of length N_2 via the DFT. However, the student **frontally** appended $N_2 - 1$ zeros to $x_1(n)$ and N_1-1 zeros to $x_2(n)$, took the DFT's of each of the resulting sequences, multiplied the spectra and then performed the IDFT. To make things worse the student inadvertently deleted $x_1(n)$ and $x_2(n)$ and was left only with the IDFT output. How can he or she obtain the desired linear convolution without doing any further DFT's or IDFT's? (10 marks)

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EXAM ID NUMBER:________

COURSE NUMBER: EE 271

PROBLEM: 1

EE Qualifying Exam

Control Theory

1. (20pts) Consider the 2nd order linear system

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where α and β are constants to be determined.

(a) (10pts) Find values of α and β so that this system *simultaneously* meets the following specifications:

- *Not* completely controllable.
- Unstable
- Stabilizable

(b) (10pts) For your values of α and β, construct state feedback, $u = Kx$, so that the closed loop system

$$\dot{x} = (A - BK)x$$

is exponentially stable.

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EXAM ID NUMBER:________

COURSE NUMBER: __EE 271

PROBLEM: ____2__

2. (20pts) Consider the 2nd order nonlinear system

$$\dot{x} = f(x, u)$$

with

$$f(x, u) = \begin{pmatrix} \sin(x_1) + x_2 \\ x_1 x_2 + u \end{pmatrix}$$

(a) (4pts) Recall that an equilibrium (x_e, u_e) is a fixed state vector

$$x_e = \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix}$$

and fixed control value u_e such that

$$f(x_e, u_e) = 0$$

Let (x_e, u_e) be an equilibrium. Suppose that the initial condition is

$$x(0) = x_e$$

and the control input satisfies

$$u(t) = u_e$$

for all t. What is the solution $x(t)$ for $t \geq 0$?

(b) (4pts) Clearly $(x_e, u_e) = (\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0)$ is an equilibrium. Construct another equilibrium in which $x_{1e} = \pi$ (i.e., $x_{2e} =?$ & $u_e =?$).

(c) (4pts) Is it possible to construct an equilibrium in which $x_{1e} = \pi/2$? If so, then construct. If not, state why not.

(d) (4pts) Linearize this system around the equilibrium $(x_e, u_e) = (\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0)$.

(e) (4pts) Is the linearization exponentially stable?