# Electrical Engineering – Written PhD Qualifier Exam
# Spring 2014

**Friday, February 7th 2014**

**Please do not write your name on this page or any other page you submit with your work. Instead use the student exam number**

**Student exam Number: ___________________________**

# Optimization
# AMCS2111

## Problem 1 (20 Points)

## LP Sensitivity [20 pts]

Consider the LP in (1), where $\epsilon$ is a small perturbation and $\mathbf{d}$ is a *known* vector.

$$\begin{aligned} \mathbf{x}^*(\epsilon) = \ & \arg\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \\ & \text{subject to: } \mathbf{A}\mathbf{x} \leq \mathbf{b} + \epsilon\mathbf{d} \end{aligned} \tag{1}$$

Take $\mathbf{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

1. **[10 pts]** Show that $\mathbf{x}^*(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Determine its active set $J$ and the optimal primal objective $p^*(0)$. You can use the following identity for invertible matrice in $\mathbb{R}^{2\times 2}$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2. **[5 pts]** Assume that $\epsilon$ is small enough that the active set of $\mathbf{x}^*(\epsilon)$ remains $J$. Express $\mathbf{x}^*(\epsilon)$ as a linear function of $\mathbf{x}^*(0)$ and $\epsilon$. What is the range of $\epsilon$ such that $\mathbf{x}^*(\epsilon)$ is still feasible?

3. **[5 pts]** Express the optimal primal objective $p^*(\epsilon)$ as a function of $p^*(0)$. What is the range of $\epsilon$ such that $\mathbf{x}^*(\epsilon)$ leads to a 50% decrease in objective as compared to $\mathbf{x}^*(0)$, i.e. such that $p^*(\epsilon) \leq \frac{1}{2}p^*(0)$?

# Optimization
# AMCS211

**Problem 2 (20 Points)**

## Principal Component Analysis [20 pts]

In this problem, we will derive a closed form solution to a traditional problem in pattern recognition, called principal component analysis (PCA).

1. To simplify the discussion of PCA, consider the constrained optimization problem in Eq (2), where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric.

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \; f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \quad \text{subject to: } \|\mathbf{x}\|_2^2 = 1 \tag{2}$$

   (i) [**3 pts**] Write down the KKT conditions for Eq (2).

   (ii) [**3 pts**] Show that $\mathbf{x}^*$ must be an eigenvector of matrix $\mathbf{A}$.

   (iii) [**4 pts**] Show that $f(\mathbf{x}^*) = \sigma_1(\mathbf{A})$, where $\sigma_1(\mathbf{A})$ is the maximum eigenvalue of $\mathbf{A}$ and $\mathbf{x}^*$ is its corresponding eigenvector.

2. Consider a set of $m$ points $\{\mathbf{y}_i\}_{i=1}^m$, where each $\mathbf{y}_i \in \mathbb{R}^n$. We seek the unit norm direction $\mathbf{w} \in \mathbb{R}^n$, along which the empirical variance of the projection of all $m$ points is *maximum*. Here, we define the projection of $\mathbf{y}_i$ along $\mathbf{w}$ as: $p_i = \mathbf{w}^T \mathbf{y}_i$. We can form the empirical variance of this projection as in Eq (3).

$$h(\mathbf{w}) = \frac{1}{m} \sum_{i=1}^{m} (p_i - \bar{p})^2, \quad \text{where } \bar{p} = \frac{1}{m} \sum_{i=1}^{m} p_i \tag{3}$$

   (i) [**6 pts**] Let $\bar{\mathbf{y}} = \frac{1}{m} \sum_{i=1}^m \mathbf{y}_i$ and $\mathbf{z}_i = \mathbf{y}_i - \bar{\mathbf{y}}$. Show that: $h(\mathbf{w}) = \mathbf{w}^T \mathbf{C} \mathbf{w}$, where $\mathbf{C} = \frac{1}{m} \mathbf{Z}\mathbf{Z}^T$ is the covariance matrix of the $m$ points and $\mathbf{z}_i$ is the $i^{\text{th}}$ column of matrix $\mathbf{Z} \in \mathbb{R}^{n \times n}$.

   (ii) [**4 pts**] PCA is the process of finding a unit norm vector $\mathbf{w}$ that maximizes the empirical projection variance $h(\mathbf{w})$. Formulate PCA as an optimization problem and use the previous results to find a closed form global solution.

# Probability and Random Processes
# AMCS241/EE241

**Problem 1 (20 Points)**

**Question 1 (10 Points)**
Let $X$ and $Y$ be independent random variables with finite variances, and let $U= X+Y$ and $V=X\,Y$. Under what condition are U and V uncorrelated ?

**Question 2 (10 Points)**
**Let $X$ be a random variable with positive entries. Consider the random variable $Y = 2a[\frac{x}{2}] - aX + 1$, where $[x]$ is the floor function of $x$ and $a$ 2 $\mathbb{R}^{\square}$. Find the probability mass function (PMF) of $Y$ and find its mean and variance if:**

$X$ follows a uniform distribution over the set $\{1, \cdots, n\}$:

$$P(X = k) = \frac{1}{n}, \; 8k \; 2 \; \{1, \cdots, n\}.$$

# Probability and Random Processes
# AMCS241/EE241

**Problem 2 (20 Points)**

**Question 1 (10 Points)**
Let X and Y have the joint probability density function (PDF) $f(x,y)= c\, x\, (y-x)\, e^{-y}$ for $0 < x < y$ and $c$ is a constant

1- **(5 Points)** Find the constant c
2- **(5 Points)** Find the conditional PDF $f(x/y)$.

**Hint:** Recall that

$$\int_{0} \sin^2(mx)\,dx = \int_{0} \text{si}$$

**Question 2 (10 Points)**
Let $Y(t)=X(t+d)-X(t)$, where $X(t)$ is a Gaussian random process.

1- **(2 Points)** Find the mean $m_Y(t)$ of Y(t) in terms of the mean $m_X(t)$ of X(t)
2- **(3 Points)** Find the auto-covariance $C_{YY}(t_1,t_2)$ of Y(t) in terms of the auto-covariance $C_{XX}(t_1,t_2)$ of X(t).
3- **(2 Points)** Find the PDF of Y(t)
4- **(3 Points)** Find the joint PDF of Y(t) and Y(t+s)

**Hint:** Recall

Chapter 5 Pairs of Random Variables

where

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}.$$

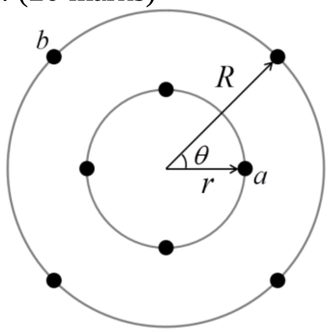
By substituting for $x$ and $y$, the argument of the exponent becomes

$$\frac{(v-w)^2/2 - 2\rho(v-w)(v+w)/2 + (v+w)^2/2}{}$$

# Digital Communications and Coding
# EE242

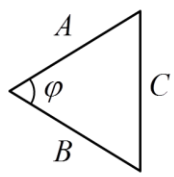
**Problem 1 (20 Points)**

Consider the constellation below which consists of two QPSK constellations offset from each other by $\pi/4$. This constellation is to be used in an AWGN channel. For parts a) to g) assume that $r=1$ and $R=\sqrt{2}$. (20 marks)




Hint: You could use the cosine law to calculate $C$ in the triangle below:

$$C^2 = A^2 + B^2 - 2AB\cos\varphi$$




a) Find the average energy of the constellation. (1 mark)

b) Determine the minimum distance. (2 marks)

c) Find the average number of nearest neighbors. (1 marks)

d) Sketch the ML decision regions for points $a$ and $b$. (4 marks)

e) Use the ML decision region to find a union bound for the probability of error given that $a$ is transmitted in terms of $E_b/N_0$ (4 marks)

f) Find a sensible way to assign bits to the various symbols. (2 marks)

g) Calculate the spectral efficiency of the constellation. (2 marks)

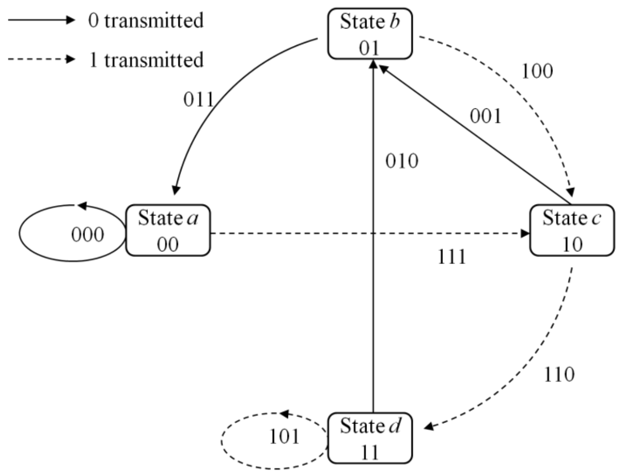
h) For $r < R$, how could you choose $r$ (with fixed $R$), so as to optimize the power efficiency of the constellation. (4 marks)

# Digital Communications and Coding
# EE242

**Problem 2(20 Points)**

**Part I (Convolution Coding)** (7 marks)

Consider the state diagram of a convolution code with rate 1/3.




The coded sequence is transmitted using BPSK in an AWGN channel such that the input to the decoder is

$$r_i = (2b_i - 1) + n_i$$

where $b_i$ is the transmitted bit and $n_i$ is the AWGN with variance $\sigma_n^2 = 0.01$.

The encoder receives the following sequence of $r_i$'s

| $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ | $r_8$ | $r_9$ | $r_{10}$ | $r_{11}$ | $r_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.9 | 2.0 | −1.3 | 0.3 | 0.8 | −0.7 | 0.1 | −2.3 | 1.7 | 0.3 | −0.02 | 1.8 |

For this sequence determine which of the following information sequences is more probable for a MAP receiver? How would things change for an ML receiver?

Seq. 1: 0 1 0 0

Seq. 2: 1 0 1 0

**Part II (Block Coding)** (13 marks)

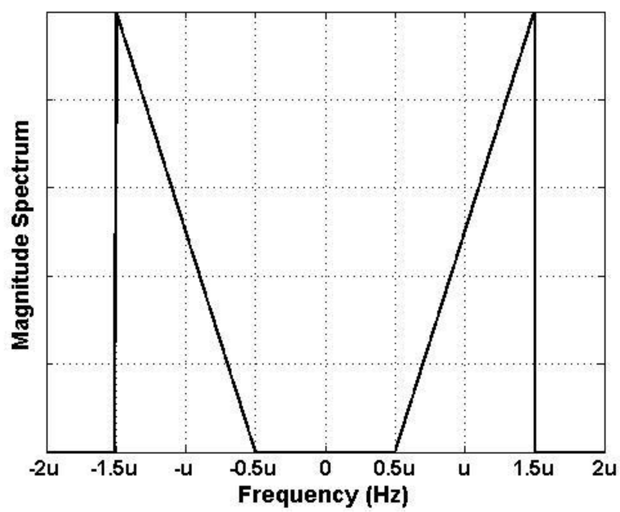
Consider the generator matrix for a systematic (6,3) block code

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

a) Determine the minimum distance of the code. (3 marks)
b) Determine the maximum number of bit errors that this code can detect. (2 marks)
c) Determine the maximum number of bit errors that this code can correct. (2 marks)
d) Determine the number of detectable error sequences. (2 marks)
e) Is $q = [1\ 0\ 1\ 1\ 0\ 0]$ a valid codeword? Why? Find the corresponding syndrome. (4 marks)

# Digital Signal Processing
# EE 251

**Problem 1 (20 Points)**




Consider the real continuous-time bandpass signal with the magnitude spectrum shown above, **u** > 0. To this signal, we apply the following operations:

1. The signal is sampled with a sampling frequency of 4**u** samples per second.
2. The signal is bandpass filtered with a filter with a passband extending from 0.5**u** to 1.5**u** Hz. Assume that the filter is perfect and note that the filter suppresses all other frequency components including those from -1.5**u** to -0.5**u** Hz.
3. The signal is downsampled by a factor of 4, i.e., **M**=4.

a- Plot the magnitude spectrum of the resulting signal after each step. For the required three plots, you should use a frequency axis extending from -6**u** to 6**u** Hz. (Each plot is worth 5 marks)
b- If we are to apply the theory of bandpass sampling to the signal above, what is the minimum sampling frequency that should be used? (5 marks)

# Digital Signal Processing
# EE 251

**Problem 2 (20 Points)**

**(a)** Calculate the DFT of the sequence $\{1,0,0,1\}$ by means of the FFT algorithm. Show the butterfly flowgraph of the FFT. (8 marks)

Let the output be $X(0), X(1), X(2)$ and $X(3)$ corresponding to the input $x(0) = 1, x(1) = 0, x(2) = 0$ and $x(3) = 1$.

**(b)** Consider the DFT output you obtained in **(a).** Assume that seven zeros are inserted after each output. In other words, we have the sequence:
$\{X(0), 0,0,0,0,0,0,0, X(1), 0,0,0,0,0,0,0, X(2), 0,0,0,0,0,0,0, X(3), 0,0,0,0,0,0,0\}$.
If we IDFT the resulting sequence of 32 samples, what is the output? Provide a numerical value for each output from $n = 0$ to 31. Write down the output as a sequence of 32 numbers separated by commas. (6 marks)

**(c)** Again consider the DFT output obtained in **(a).** Assume that the output is repeated **eight** times. That is, we have the sequence:
$\{X(0), X(1), X(2), X(3), X(0), X(1), X(2), X(3), X(0), X(1), X(2), X(3), \ldots X(2), X(3)\}$
If we IDFT the resulting sequence of 32 samples, what is the output? Provide a numerical value for each output from $n = 0$ to 31. Write down the output as a sequence of 32 numbers separated by commas. (6 marks)

# Communication Networks
# EE262

**<u>Problem 1 (Network and Internet) 20 Marks</u>**

Consider sending a file $F$ bits over a path of $Q$ links. Each link transmits at $R$ bps. The network is lightly loaded such that there are no queuing delays and propagation delay is negligible. When packet switching is used, the file $F$ bits are broken up into $M$ packets, each packet with $L$ bits. *(20 Marks, 4 Marks each)*

a. How long does it take to send the file $F$ over the path with $Q$ links?

b. Suppose the network is circuit switched. The time required to set the path is $T$ seconds. For each packet the sending layers add a total of $h$ bits of header. How long does it take to send the file from source to destination?

c. Suppose the network is a packet-switched and a connectionless service is used. Also, suppose each packet has $2h$ bits of header. How long does it take to send the file?

d. Suppose the network is a packet-switched and a connection-oriented service is used. Also, suppose each packet has $h$ bits of header. A total of 3 extra packets are sent prior the actual file is transmitted to establish the connection (TCP handshaking). How long does it take to send the file?

e. Suppose that the network is a circuit switched with transmission rate of $R$ bps. Assuming $T$ set-up time and $h$ bits of header appended to the entire file, how long does it take to send the file?

# Communication Networks
# EE262

**<u>Problem 2 (*Transport Layer*) 20 Marks</u>**

In TCP over heterogeneous networks *(20 Marks, 4 Marks each)*

a) Consider applications that transmit data at a constant rate. Suppose that a packet-switched network is used and the only traffic in this network comes from such applications. The sum of the application data rates is less that the capacities of each and every link. Is some form of congestion control needed? Justify your answer.

b) List the TCP congestion control phases and explain the purpose of each phase.

c) List one problem induced when delay-based TCPs run over Optical Burst Switched Networks

d) Name one approach that assists the problem of running dropping-based TCP over Optical Burst Switched networks

e) Repeat (c) by replacing optical burst switched links with wireless links

# Control Theory
# EE271 A

**Problem 1 (20 Points)**

The dynamics of a nonlinear system is $C\,\dot{v}(t) + (z(t))^2 = -\mu v(t) + u(t)$ and $\dot{z}(t) = -\sin(v(t))$, where v(t) and z(t) are two scalar functions, C and $\mu$ are given positive constants and u(t) is a control.

1. Write the nonlinear state space equation of the system, in standard form. What is the order of the system? (/4)
2. Find the set of possible equilibria, as a function of u (which is assumed to be constant for this question) (/3)
3. What equilibrium input must be applied to stabilize the system around v=0 and z=1? (/2)
3. Linearize the system around the equilibrium position (v=0 and z=1) and write the dynamical equations of the linearized system in state space form (/5)
4. Is the linearized system stable? (/4)
5. True or false: if a linearized system is stable, then the original nonlinear system is also stable (completely and globally). If true, show it, if false find a counterexample. (/2)

# Control Theory
# EE271 A

**Problem 2 (20 Points)**

**Part 1:**

We consider a linear system of the form $\dot{x} = Ax + Bu$ and $y = Cx$.

1. Assume that A is of dimension n, and B is a n by 1 vector (single input). Can the system be completely controllable? If yes, give a general example (with A of size n, i.e. create a n by n A matrix and an n by 1 vector B for which the above system is completely controllable), if no prove it. (/3)

We now assume that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$ and $C = (0 \quad 1)$.

2. Show that this system is completely controllable (/2)
3. Design a state feedback controller such that the closed loop system is stable, and has eigenvalues -2,-2. Give the gain matrix. Is the gain matrix solution to this problem unique? (/5)

**Part 2:**

4. Recall the definition of the stability of a system $\dot{x} = Ax + Bu$, when no control input is present. How do you check stability for a linear time invariant system in practice? (/4)
5. Now, assume that we have a discrete time system x(k+1)=Ax(k), y(k)=Cx(k). Show using the definition of stability of a system that this system is stable if and only if the norm of the matrix A^k is bounded for all k (/3).
6. Show using the spectral mapping theorem that if the eigenvalues of a matrix A are all strictly lower than one (in modulus), then the matrix A^k has bounded eigenvalues (/3). Recall: the spectral mapping theorem states that if s is an eigenvalue of A, then f(s) is an eigenvalue of f(A).