Question #1 (AMCS211) [20pts]

Using KKT Conditions. Consider the unit ℓ_2 ball constrained linear least squares problem in (1). Here, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a full column rank matrix and has unit singular values, *i.e.* $\sigma_i(\mathbf{A}) = 1\ \forall i = 1, \dots, n$.

$$\begin{aligned} \mathbf{x}^* = \ & \arg\min_{\mathbf{x}} \ \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 \\ & \text{subject to: } \ \|\mathbf{x}\|_2^2 \leq 1 \end{aligned} \tag{1}$$

(a) Is (1) convex? Explain. Write down the Lagrangian and KKT conditions of (1). **[5pts]**

(b) Use the stationary Lagrangian KKT condition above to show that the solution is proportional to $\mathbf{A}^\top\mathbf{b}$. Explain why it is a global solution. **[5pts]**

(c) Use the other KKT conditions to derive the expression in (2). DO NOT simply check if (2) satisfies these conditions. **[10pts]**

$$\mathbf{x}^* = \min\left(1, \frac{1}{\|\mathbf{A}^\top\mathbf{b}\|_2}\right)\mathbf{A}^\top\mathbf{b} \tag{2}$$

Question #2 (AMCS211) [20pts]

Solving a 2D LP. In this problem, we will study the LP in (3).

$$(x_1^*, x_2^*) = \underset{x_1, x_2}{\arg\max}\ f(x_1, x_2) = \frac{x_1}{2} + \frac{x_2}{2} \tag{3}$$

$$\text{subject to: } \begin{cases} 0 \leq x_1 \leq 2 \\ 0 \leq x_2 \leq 2 \\ x_1 + x_2 \geq 1 \end{cases}$$

(a) Graph the feasible constraint set of the LP. **[5pts]**

(b) Graphically, compute its global solution (x_1^*, x_2^*) and the optimal objective p^*. Then, show *mathematically* that it is optimal for this problem. Remember to determine its active set J. **[10pts]**

(c) Find another linear combination $f(x_1, x_2) = c_1 x_1 + c_2 x_2$, such that the *unique* optimal solution is at $(x_1^* = 1, x_2^* = 0)$. Explain and justify your answer. **[5pts]**

1. a. Consider a real-valued zero-mean Gaussian random process $X(t),\ t \geq 0$. The autocorrelation function of the process is $R_{XX}(t_1, t_2) = \min(t_1, t_2)$. Suppose that

$0 < s < t < u < v$. Obtain the conditional correlation coefficient between $X(t)$ and $X(u)$ given $X(s)$ and $X(v)$ as a function of s, t, u and v. **(10 marks)**

b. Consider real-valued random variable X with characteristic function $\phi_X(u)$. Prove that:

$\mathbb{P}\left(|X| > \frac{2}{T}\right) \leq 2\left(1 - \frac{1}{2T}\int_{-T}^{T} \phi_X(u)du\right)$. **(10 marks)**

2. a. A random process $X(t), t \geq 0$, satisfies: $X(0) = 0$ and $X(s) - X(t)$ is independent of $X(t)$ for all $s > t > 0$. Prove that the autocovariance function of the process is:

$C_{XX}(t_1, t_2) = \text{Var}\{X(\min(t_1, t_2))\}$. **(4 marks)**

b. Consider random process $X(t), t \geq 0$. Can the process have an autocovariance function $C_{XX}(t_1, t_2) = |t_1 - t_2|$? Justify your answer. **(4 marks)**

c. Gerald loves books. His acquisition of new books can be modeled as a Poisson process with a rate of $\frac{1}{2}$ books per week. The times of reading different books are i.i.d. **Exp** $\left(\frac{1}{3}\right)$, i.e., it takes Gerald 3 weeks on average to finish one book. The time to read any book is independent of the book acquisition process. If Gerald acquires a new book while he still did not finish his current reading, he immediately starts reading the new book and only comes back to the older book when he finishes the new book.

i. When Gerald starts a new book, what is the probability that he will finish it without being interrupted? **(3 marks)**

ii. Given that Gerald receives a new book while reading another book, what is the probability that he can finish both books, the new one and the interrupted one, without further interruption? **(4 marks)**

iii. What is the average reading time of a book given that it is not interrupted? **(5 marks)**

KING ABDULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY
ELECTRICAL ENGINEERING DEPARTMENT
EE 242 Digital Communications and Coding
Qualification Exam
May 29, 2018
(*Time: 3 hours*)

Q1: Consider the 2-user BPSK signaling in AWGN where the received signal is given by

$$\mathbf{y} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \mathbf{n}$$

where $\mathbf{u}_1 = (-1, -1)^T$ and $\mathbf{u}_2 = (2, 1)^T$ and $b_i = \pm 1$ is the bit transmitted by user i (i=1 or 2). The noise $\mathbf{n}$ is zero mean white Gaussian noise with covariance $\frac{N0}{2}\mathbf{I}$ with $\frac{N0}{2} = 0.001$.

To solve this problem, consider this as M-ary signaling with M =4.

a. (3 marks) Draw the signal constellation in the (y_1, y_2) plane.

b. (3 marks) Specify the MAP decision for $\mathbf{y} = (2.5, 1)^T$.

c. (5 marks) Repeat part b for the ML and maximum correlation receivers.

d. (9 marks) Determine the minimum distance for the constellation and the intelligent union bound.

Q2: Consider the following (5, 3) block code defined by the Table 3.

Information sequence	Codeword
000	00000
001	01001
010	11010
100	10010
011	11100
101	10101
110	00110
111	01111

Table 1: Table for Q4

The left column is the information sequence and the right column is the corresponding codeword. The information source emits 1's with probability 0.6 and zeros with probability 0.4.

a. (2 marks) Is this a linear code ? Explain.

b. (2 marks) Find the minimum weight of the code.

c. (2 marks) By inspection, find the minimum distance of the code.

d. (4 marks) Determine the error correction and error detection capability of the code.

e. (6 marks) The communication channel is a BSC with crossover probability $p = 0.1$. At the output of this channel, we received the following sequence $y = 1\,0\,0\,0\,1$. Determine the ML estimate of the information sequence.

f. (4 marks) Repeat part (e) and find the MAP estimate of the information sequence.

1. a. Consider real-valued bandlimited signal $x(t)$ with bandwidth $W \in \mathbb{R}^+$ and the signal $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT)$, where $T = \frac{1.25}{W}$. Show that $y(t)$ can be expressed as:

$$y(t) = A + B\cos(Dt + E)$$

The constants A, B, D and E should be provided in terms of $\text{CTFT}\{x(t)\}$, i.e., the continuous-time Fourier transform of signal $x(t)$. **(10 marks)**

b. In this problem we denote by $z_{\text{LPF},f_c}(t)$ the signal that results from filtering energy signal $z(t)$ using the low-pass filter (LPF) with frequency response **rect** $\left(\frac{f}{2f_c}\right)$ (i.e., the LPF has a cutoff frequency equal to f_c Hz). Consider energy signals $x(t)$ and $y(t)$. Moreover, signal $x(t)$ is bandlimited and has a bandwidth of W Hz. Assuming that $f_c > W$, prove that:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = C \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_c}\right) y^*_{\text{LPF},f_c}\left(\frac{n}{2f_c}\right)$$

for some positive constant C. (The asterisk denotes complex conjugation.) What is the value of C in terms of W and f_c? **(10 marks)**

2. a. Determine a discrete-time sequence $x(n)$ that satisfies all of the following 3 conditions:

i. The discrete-time Fourier transform (DTFT) of $x(n)$ has the form:

$$1 + \beta_1 \cos(2\pi f T_s) + \beta_2 \cos(4\pi f T_s)$$

ii. The sequence $x(n) * \delta(n-3)$ evaluated at $n = 2$ is 5. (The asterisk $*$ denotes convolution and $\delta(n)$ is the discrete-time delta.)

iii. For the 3-point sequence $y(n)$ such that $y(0) = 1$, $y(1) = 2$ and $y(3) = 3$, the result of the 8-point circular convolution of $y(n)$ and $x(n-3)$ is 11 when $n = 2$. **(4 marks)**

b. Obtain the eigenvalues of the following matrix. **(4 marks)**

$$Y = \begin{bmatrix} 1 & 1-\frac{1}{2}i & -2 & 1+\frac{1}{2}i \\ 1+\frac{1}{2}i & 1 & 1-\frac{1}{2}i & -2 \\ -2 & 1+\frac{1}{2}i & 1 & 1-\frac{1}{2}i \\ 1-\frac{1}{2}i & -2 & 1+\frac{1}{2}i & 1 \end{bmatrix}$$

c. Consider positive integers D, N and M, the discrete-time sequence $\{x(n)\}_{n=0}^{NM-1}$ of length NM, and integer a such that a and NM are relatively prime.

i. Let $X(k)$ be the NM-point discrete Fourier transform (DFT) of $x(n)$ and let $Y(k) = X(ak)$. $Y(k)$ is downsampled by M and the N-point inverse DFT (IDFT) of the result is computed to obtain $z(n)$. Express sequence z in terms of sequence x. **(4 marks)**

ii. If M is even and $x(n) = \exp\left(-\frac{i\pi n^2}{N}\right)$, $n = 0, 1, \dots, NM-1$. Find $X(k)$. Note that the D-point DFT of $\exp\left(-\frac{i2\pi n^2}{D}\right)$ is $\sqrt{\frac{-iD}{2}}\left[1 + i^D(-1)^k\right]\exp\left(\frac{i\pi}{2D}k^2\right)$. **(4 marks)**

iii. If $N = 38, M = 2, a = 21$ and $x(n) = \exp\left(-\frac{i\pi n^2}{N}\right)$, compute $Y(3)$ and $z(19)$ Note that $a^{-1} = 29 \bmod 76$. ("mod" is the modulo operation, which finds the remainder after division of one number by another.) **(4 marks)**

EE Qualifying Exam

Control Theory

1. Consider the linear system

$$\begin{aligned}\dot{x} &= Ax \\ &= \begin{pmatrix} -1 & 0 \\ \alpha & -2 \end{pmatrix} x\end{aligned}$$

where the value of α is unspecified.

(a) What are the eigenvalues and eigenvectors of A? (as a function of α)

(b) For what values of α is the system stable?

(c) Construct the matrix exponential in the form

$$e^{At} = Ve^{\Lambda t}V^{-1}$$

for appropriate V and diagonal Λ (both of which might depend on α).

(d) Now let $\alpha = 1$. Construct a Lyapunov function

$$V(x) = x^{\mathrm{T}}Px$$

that certifies that the system is stable.

2. Consider the linear system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u\end{aligned}$$

(a) Is this system controllable?

(b) What is the **definition** of controllable?

(c) For what values of γ is the feedback

$$u = -\gamma \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

stabilizing?

(d) Now define the output

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Construct a output feedback controller,

$$\begin{aligned}\dot{z} &= Ez + Fy \\ u &= Gz\end{aligned}$$

such that the overall closed-loop system is stable (i.e., construct suitable E, F, G).